COMBINATORICA 2 (2) (1982) 103—109

SYMMETRIC DESIGNS WITH BRUCK SUBDESIGNS

R. D. BAKER

Received 2 September 1980

If P is a finite projective plane of order n with a proper subplane Q of order m which is not a Baer subplane, then a theorem of Bruck [Trans. AMS 78(1955), 464—481] asserts that $n \geqq m^2+m$. If the equality $n=m^2+m$ were to occur then P would be of composite order and Q should be called a Bruck subplane. It can be shown that if a projective plane P contains a Bruck subplane Q, then in fact P contains a design Q' which has the parameters of the lines in a three dimensional projective geometry of order m. A well known scheme of Bruck suggests using such a Q' to construct P. Bruck's theorem readily extends to symmetric designs [Kantor, Trans. AMS 146 (1969), 1—28], hence the concept of a Bruck subdesign. This paper develops the analoque of Q' and shows (by example) that the analogous construction scheme can be used to find symmetric designs.

1. Introduction and Summary

If P is a finite projective plane of order n with a proper subplane Q of order m, then a theorem of Bruck [3] states that either $n=m^2$ or $n \geqq m^2+m$. The equality $n=m^2$ occurs if and only if every point of P is on a line of Q (and dually), and in this case Q is called a Baer subplane of P. The equality $n=m^2+m$ would mean that the order of P is not a prime power, and naturally Q ought to be called a Bruck subplane of P. Bruck's theorem is easily extended to symmetric designs (see Kantor [6]). Bruck [4] outlines a construction scheme for a projective plane P of order $n=m^2+m$ based on extending the design given by the points and lines of a projective 3-space of order m. If such a plane P were constructed, then each subplane Q of the 3-space would be a Bruck subplane of P. After presenting some basic notation and terminology in the second section, this paper investigates the general structure of a symmetric design which contains a Bruck subdesign. It is shown that there must exist a design analogous to the projective 3-space in Bruck's construction scheme. Some other substructures are also listed, then in the following section two construction schemes based on this information are given. The last section gives some exaples to illustrate how these schemes can be applied.

AMS (1980) subject classification: 05 B 30, 05 B 25; 50 D 35, 62 K 10.

2. Basic Notation and Terminology

An incidence structure S is a pair of (finite) sets X and $\mathscr{B}$ and an incidence relation between them. Elements of X are called points and elements of $\mathscr{B}$ are called blocks. S is regular with replication number r if each point is incident with exactly r blocks. S is uniform with block size k if each block is incident with exactly k points. A block will be identified with its subset of incident points even though two distinct blocks may thus be the same subset of X, and incidence is then set theoretic inclusion. The cardinality of X is denoted by v and the cardinality of $\mathscr{B}$ by b. For a regular, uniform incidence structure $bk=vr$. A substructure S_0 of S is a pair $(X_0, \mathscr{B}_0)$ of subsets of X and $\mathscr{B}$, respectively, with incidence the restriction of incidence in S. A tactical decomposition of S is a partition of X into disjoint classes $X_1, X_2, \ldots, X_d$ and a partition of $\mathscr{B}$ into disjoint classes $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_c$ such that every pair $(X_i, \mathscr{B}_j)$ is a regular, uniform substructure, with l_i points, m_j blocks, block size β_{ij}, and replication number γ_{ij}. A regular uniform incidence structure S is symmetric if $b=v$, or equivalently $k=r$.

A design D is a uniform incidence structure with block size k such that each pair of distinct points are simultaneously incident with exactly λ blocks. It follows that D is regular with replication number r and $r(k-1)=\lambda(v-1)$. D is also called a (v, k, λ)-design, is denoted by $B_\lambda[k, v]$, and is said to have parameters (v, k, λ). A symmetric design is a design whose incidence structure is symmetric, and is denoted by $SB_\lambda[k, v]$. When $\lambda=1$ it is suppressed from either of the notations $B_\lambda[k, v]$ or $SB_\lambda[k, v]$. A $SB[k, v]$ is a projective plane, $n=k-1$ is its order and $v=n^2+n+1$. A subdesign of a design is a substructure of the incidence structure which is also a design. A proper subdesign D_0 with parameters (v_0, k_0, λ) of a $SB_\lambda[k, v]$ is called a Baer subdesign if $k-\lambda=(k_0-1)^2$ and is called a Bruck subdesign if $k-\lambda=k_0(k_0-1)$.

For the remainder of this paper all incidence structures are assumed to be regular and uniform unless otherwise stated.

3. Structure Theory

An analysis of the structure of a symmetric design which contains a Bruck subdesign must begin with the theorem which inspired the name.

Theorem 1. *If D is a $SB_\lambda[k, v]$ which contains D_0, a $SB_\lambda[k_0, v_0]$, then $k=\lambda v_0-k_0+1$ or $k\geqq\lambda v_0$.*

Proof. See Kantor [6] for the theorem as stated or Bruck [3] for the special case $\lambda=1$, the proofs being essentially the same. An alternate form for the conclusion is $k-\lambda=(k_0-1)^2$ or $k-\lambda\geqq k_0(k_0-1)$. ∎

Corollary 2. *If D_0 is a $SB_\lambda[k_0, v_0]$ contained in D a $SB_\lambda[k, v]$, then each block of D not in D_0 contains at most one point of D_0 and dually each point of D not in D_0 is incident with at most one block on D_0. Moreover in each case the 'at most' may be changed to 'exactly' if and only if D_0 is a Baer subdesign.*

Proof. The 'at most' part of the argument follows directly from the definition of a subdesign, and the exactly arises in the proof of Theorem 1. ∎

Theorem 3. *If D is a $SB_\lambda[k, v]$ which contains a Bruck subdesign $D_0=(X_0, \mathscr{B}_0)$, then D admits a tactical decomposition with point classes X_0, X_1, X_2 and block classes $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2$, and having parameters as follows:*

$$(1) \qquad (l_0, l_1, l_2) = (m_0, m_1, m_2) = \big(v_0, v_0(k_0-2)+1, v_0(k-k_0)\big),$$

$$(2) \qquad (\beta_{ij})^t = (\gamma_{ij}) = \begin{pmatrix} k_0 & 0 & k-k_0 \\ 0 & 0 & k \\ 1 & k_0-1 & k-k_0 \end{pmatrix}$$

where $(\beta_{ij})^t$ denotes the transpose of (β_{ij}).

Proof. D_0 is $(X_0, \mathscr{B}_0)$. X_1 is the set of points not incident with any block of D_0 and $\mathscr{B}_1$ is the set of blocks not incident with any point of D_0. X_2 and $\mathscr{B}_2$ are the remaining points and blocks, which are nonempty by Corollary 2. Counting and the equation $k=\lambda v_0$ produce the expressions for the cardinalities of the point and block classes. To show (2) it suffices to consider either the block sizes or the replication numbers of the substructures by duality, and hence only block sizes are determined. Corollary 2 and the definitions of the parts show that the first row and the first column are correct. Also the entries in each row must sum to k, thus only two more numbers need to be computed. For β_{22} one notes that for a fixed block of $\mathscr{B}_0$ the intersections with the blocks of $\mathscr{B}_0$ partition its points in X_2 into k_0 sets of size $(\lambda-1)$ (from the blocks on the unique point of X_0 in the fixed block) and v_0-k_0 sets of size λ. Hence $\beta_{22}=(\lambda-1)k_0+\lambda(v_0-k_0)=k-k_0$. Similarly β_{21} is $\lambda v_0=k$. From a result for tactical decompositions on the matrix product $(\gamma_{ij})(\beta_{ij})^t$ (see Dembowski [5]) it follows that

$$(\beta_{ij})^2 = (k-\lambda)\, I + \lambda \begin{pmatrix} l_1 & l_1 & l_1 \\ l_2 & l_2 & l_2 \\ l_3 & l_3 & l_3 \end{pmatrix}. \ \blacksquare$$

Corollary 4. *If D is a $SB_\lambda[k, v]$ which contains a Bruck subdesign D_0, then D contains a subdesign D_1 which in turn contains D_0, where D_1 is a $B_\lambda[k_1, v_1]$ with $v_1=v_0(k_0-1)+1$ and $k_1=k_0$.*

Proof. The points of D_1 are $X_0 \cup X_1$ and the blocks are $\mathscr{B}_0 \cup \mathscr{B}_2$. The parameters are computed from Theorem 3, with the design property for pairs of points inherited from D since blocks of $\mathscr{B}_1$ are disjoint from the set of points in D_1.

One easily computes that for D_1 in Corollary 4 there are $b_1=v_0(k-k_0+1)$ blocks and each point is in $r_1=k$ blocks. It is an open problem whether there exists a projective plane P of order n which contains a Bruck subplane Q or order m. If there exists such a plane P, then P contains D_1 which is a $B[m+1, m^3+m^2+m+1]$. Thus D_1 would have the same parameters as a design whose points and blocks are the points and lines of a projective 3-space of order m. It is an open problem whether D_1 would have to be such a 3-space. However, if it were then Q would be a Desarguesian plane and, in fact, P would contain many Bruck subplanes. $\blacksquare$

Corollary 5. *If D is a $SB_\lambda[k, v]$ which contains a Bruck subdesign D_0, then D contains a symmetric incidence substructure C which has $v_0(k-k_0)$ points and blocks of size $k-k_0$. Moreover C admits a tactical decomposition which has v_0 point* *classes each of size* $k-k_0$, v_0 *block classes each of size* $k-k_0$, *and the* β_{ij} *and* γ_{ij} *equal to either* λ *or* $\lambda-1$.

Proof. C is the pair $(X_2, \mathscr{B}_2)$ in the notation of Theorem 3. The point classes of the tactical decomposition are the blocks of $\mathscr{B}_0$ and the block classes are the points of X_0 considered as sets of blocks. ∎

4. Construction Schemes

Common methods for constructing designs include direct constructions which usually employ some automorphisms of the design to reduce computations (differences methods), composition methods which amalgamate many 'small' designs and/or incidence structures to obtain a design on a 'large' number of points, and extension of a design or incidence structure to obtain a design with the desired parameters. The two schemes given below are for extension of the design D_1 of Corollary 4 and for extension of the incidence structure C of Corollary 5. As in all extension schemes some concept is needed to 'find' the missing points and blocks, and their incidences.

A spread of a design or incidence structure is a collection of blocks such that each point is in exactly λ blocks of the spread. In other words a spread is a collection of blocks such that the substructure given by all points and these blocks is regular. A packing is a collection of spreads such that each block is in exactly λ spreads of the packing. A spread is incident with each block it contains and with each packing which contains it. A packing is incident with each spread it contains. Spreads are to be some of the missing points of the desired symmetric design, and hence will be refered to as 'new points'. Likewise packing will be refered to as 'new blocks'. The first construction is the generalization of Bruck's construction, and only requires finding suitable collections of spreads and packings.

Construction D. *Given a* $B_\lambda[k_1; v_1]D_1$ *which contains a* $SB_\lambda[k_0, v_0]D_0$ *such that* $r_1=\lambda v_0$ *and* $k_1=k_0$, D_1 *can be extended to a* $SB_\lambda[k, v]D$ *with* $k=\lambda v_0$ *if and only if there exists a collection* Y *of* $v_0(k-k_0)$ *spreads and a collection* $\mathscr{A}$ *of* $v_0(k_0-2)+1$ *packings which satisfy:*

(1) *Each block is in* $k-k_0$ *spreads and*

(2) *the number of blocks plus the number of packings two spreads have in common is* λ.

Verification. Adjoining the spreads as new points and the packings as new blocks with incidence as described above gives the design. Since two of the original points are in exactly λ blocks (which are in D_1), an old and a new point (spread) are in exactly λ blocks by the definition of a spread, and condition (2) gives the same property for two new points, the new structure is a design if all blocks have size k. Each old block has size k by condition (1), namely k_0 old and $k-k_0$ new points, and the definition of a packing implies that each new block will contain $\lambda v_0=k$ new points. The reverse implication follows from Corollary 4 with $Y=X_2$ and $\mathscr{A}=\mathscr{B}_1$. ∎

In the next construction the point and block classes of a tactical decomposition will be used as new blocks and points respectively. Some new blocks need to be found as subsets of points, and spreads will be used as new points. The new blocks are to be dual-spreads, i.e. each block must contain exactly λ points of such a new block. The equations which must be satisfied by the parameters of a symmetric incidence structure such as given in Corollary 5 are nontrivial, hence a definition precedes the statement of Construction C.

Definition. A symmetric incidence structure $C=(X_2, \mathscr{B}_2)$ with hm points and blocks of size m is called *admissible* for extension to a symmetric design with a Bruck subdesign if

(1) $k_0 = \dfrac{\sqrt{4h(h-1)(m+1)+1}+(2h-1)}{2h}$ is a positive integer,

(2) any two distinct points are contained in at most $\lambda = m-(k_0-1)^2+1$ blocks, and

(3) $m \leqq h^2-h$.

The Bruck subdesign would be a $SB_\lambda[k_0, v_0]$ where $v_0=h$ and $k=m+k_0$ would be the block size of the symmetric design.

The above definition is formulated by equating m to $k-k_0$ and hm to $v_0(k-k_0)$ to match Corollary 5. It may be verified that the resulting parameters satisfy the relations for a symmetric design and $\lambda h=k$. The inequality (3) is equivalent to $k_0 \leqq v_0$.

Construction C. *An admissible symmetrice incidence structure $C=(X_2, \mathscr{B}_2)$ on hm points with blocks of size m can be extended to a $SB_\lambda[k, v]$ with a Bruck subdesign if C admits a tactical decomposition into h point and h block classes each of size m, there exists a collection Y of $h(k_0-2)+1$ spreads, and a collection $\mathscr{A}$ of $h(k_0-2)+1$ dual-spreads which satisfy:*

(1) *Each block is in k_0-1 spreads,*

(2) *each spread contains exactly λ blocks from each block class,*

(3) *any two spreads have exactly λ blocks in common,*

(4) *two points from the same point class are contained in $\lambda-1$ blocks of $\mathscr{A} \cup \mathscr{B}_2$,*

(5) *two points from different point classes are contained in λ blocks of $\mathscr{A} \cup \mathscr{B}_2$,*

(6) *the values of the γ_{ij} are λ or $\lambda-1$,*

(7) *two distinct block classes have exactly λ point classes which simultaneously yield $\gamma_{ij}=\lambda-1$.*

Verification. First the incidence structure for the design is defined. Second the block sizes are computed. Finally the design property for pairs of points is checked.

The points of the design are X_2, Y, and the block classes of the tactical decomposition. The blocks are $\mathscr{B}_2$, $\mathscr{A}$, and the point classes of the tactical decomposition. Both the blocks of $\mathscr{A}$ and the spreads of Y are only incident with the elements of C which they contain. The point classes and the block classes are incident with the points and blocks they contain. Also the i-th point class and the j-th block class are incident if $\gamma_{ij}=\lambda-1$.

Each block of $\mathscr{B}_2$ contains $m=k-k_0$ points of X_2, is in k_0-1 spreads of Y by (1), and is in a unique block class to net a total of k points. Since each block of $\mathscr{A}$ is a dual-spread it contains $\lambda hm/m=\lambda h$ points, and $\lambda h=k$ by the definition of admissible. From condition (6) together with the fact that $\gamma_{ij}+\gamma_{i2}+\ldots+\gamma_{ih}=k-k_0$ it follows that a point class is incident with exactly k_0 block classes in addition to its $k-k_0$ points. Hence all blocks will have size k.

For two points of X_2 conditions (4) and (5) show that they are contained in exactly λ blocks one of which might be a point class. A point of X_2 and a spread have λ blocks in common by the definition of a spread. A point of X_2 and a block class have exactly λ blocks in common by condition (6) where one block might be a point class. Two spreads have λ blocks in common by condition (3). A spread and a block class have λ blocks in common by condition (2). Finally condition (7) yields the result for two block classes. ∎

5. Applications

Both Construction D and Construction C may be used to construct symmetric designs provided there exist designs which contain a Bruck subdesign. The theorem which follows asserts that at least one family of such designs exist. A specific example is then given to illustrate the constructions.

Theorem 6. *If n and $n-1$ are both prime powers, then there exists a $SB_n[n^2, n^3-n+1]$ which contains a $SB_n[n, n]$.*

Proof: Wilson [7] gives a direct construction for a $SB_n[n^2, n^3-n+1]$ using as the point set the union of n affine planes of order n having a unique point in common (so the number of points is $1+n(n^2-1)$). Beker and Piper [2] use $n+1$ disjoint affine planes of order $n-1$ to construct the residual of a design with the same parameters. Wilson's construction is quite general since it allows the selection at certain stages of any design with a given set of parameters on a given set of "points". If at several of these stages the auxiliary design is required to contain a particular block, then the resulting $SB_n[n^2, n^3-n+1]$ will contain a $SB_n[n, n]$. ∎

Example of Construction D. Consider $b=25, k=9$, and $\lambda=3$, so that $v_0=k_0=3$ and $v_1=7, k_1=3$. For the points of the $B_3[3, 7]$ take the integers modulo 7. For the blocks take $B_{i,j}=\{1+i, 2+i, 4+i\}$ with $i=0, 1, \ldots, 6$, and $j=0, 1, 2$. Thus for each i the points of $B_{i,0}$ and the three blocks $B_{i,0}, B_{i,1}, B_{i,2}$ form a $SB_3[3, 3]$. The construction requires 18 spreads and 4 packings. The spreads are:

$$\mathscr{S}_{\infty,j} = \{B_{0,0}, B_{1,j}, B_{2,j}, B_{3,j}, B_{4,j}\, B_{5,j}, B_{6,j}\},$$

$$\mathscr{S}_{0,j} = \{B_{0,0}, B_{1,j}, B_{2,j}\, B_{3,j+1}, B_{4,j+2}, B_{5,j+2}, B_{6,j+1}\},$$

$$\mathscr{S}_{1,j} = \{B_{0,1}, B_{1,j}, B_{2,j+1}, B_{3,j}, B_{4,j+1}, B_{5,j+2}, B_{6,j+2}\},$$

$$\mathscr{S}_{2,j} = \{B_{0,2}, B_{1,j}, B_{2,j+2}, B_{3,j+1}, B_{4,j}, B_{5,j+1}, B_{6,j+2}\},$$

$$\mathscr{S}_{3,j} = \{B_{0,2}, B_{1,j}, B_{2,j+2}, B_{3,j+2}, B_{4,j+1}, B_{5,j}, B_{6,j+1}\},$$

$$\mathscr{S}_{4,j} = \{B_{0,1}, B_{1,j}, B_{2,j+1}, B_{3,j+2}, B_{4,j+2}, B_{5,j+1}, B_{6,j}\},$$

for $j=0, 1$, and 2 (the second subscript read modulo three). The packings are:

$$\mathscr{P}_1 = \{\mathscr{S}_{\infty,0}, \mathscr{S}_{\infty,1}, \mathscr{S}_{\infty,2}, \mathscr{S}_{1,0}, \mathscr{S}_{1,1}, \mathscr{S}_{1,2}, \mathscr{S}_{2,0}, \mathscr{S}_{2,1}, \mathscr{S}_{2,2}\},$$

$$\mathscr{P}_2 = \{\mathscr{S}_{\infty,0}, \mathscr{S}_{\infty,1}, \mathscr{S}_{\infty,2}, \mathscr{S}_{4,0}, \mathscr{S}_{4,1}, \mathscr{S}_{4,2}, \mathscr{S}_{3,0}, \mathscr{S}_{3,1}, \mathscr{S}_{3,2}\},$$

$$\mathscr{P}_3 = \{\mathscr{S}_{0,0}, \mathscr{S}_{0,1}, \mathscr{S}_{0,2}, \mathscr{S}_{1,0}, \mathscr{S}_{1,1}, \mathscr{S}_{1,2}, \mathscr{S}_{3,0}, \mathscr{S}_{3,1}, \mathscr{S}_{3,2}\},$$

$$\mathscr{P}_4 = \{\mathscr{S}_{0,0}, \mathscr{S}_{0,1}, \mathscr{S}_{0,2}, \mathscr{S}_{4,0}, \mathscr{S}_{4,1}, \mathscr{S}_{4,2}, \mathscr{S}_{2,0}, \mathscr{S}_{2,1}, \mathscr{S}_{2,2}\}.$$

Example of Construction C. Again consider $v=25$, $k=9$, and $\lambda=3$. The construction begins with a symmetric incidence structure C with 18 points and blocks of size 6. In fact the C given is a "reasolvable symmetrical semi-regular GD design". The points of C are $(Z_5\cup\{\theta\})\times Z_3$, where Z_s denotes the integers modulo s. The blocks are $B_{\infty,j}=\{(\theta,j),(0,j),(1,j),(2,j),(3,j),(4,j)\}$ and $B_{i,j}=\{(\theta,j),(0+i,j),(1+i,j+1),(2+i,j+2),(3+i,j+2),(4+i,j+1)\}$ for $i=0,1,\dots,4$ and $j=0,1,2$. The construction requires finding a tactical decomposition with 3 point and 3 block classes each of size 6, 4 spreads and 4 blocks of size 9 on the points of C. The point classes are $\{\theta,0\}\times Z_3$, $\{1,4\}\times Z_3$, and $\{2,3\}\times Z_3$. The block classes are $\{B_{i,j}: (i,j)\in\{\theta,0\}\times Z_3\}$, $\{B_{i,j}: (i,j)\in\{1,4\}\times Z_3\}$, and $\{B_{i,j}: (i,j)\in\{2,3\}\times Z_3\}$. The spreads are:

$$\mathscr{S}_1=\{B_{i,j}:\ (i,j)\in\{\theta,1,3\}\times Z_3\},$$
$$\mathscr{S}_2=\{B_{i,j}:\ (i,j)\in\{\theta,2,4\}\times Z_3\},$$
$$\mathscr{S}_3=\{B_{i,j}:\ (i,j)\in\{0,1,2\}\times Z_3\},\quad\text{and}$$
$$\mathscr{S}_4=\{B_{i,j}:\ (i,j)\in\{0,3,4\}\times Z_3\}.$$

The new blocks of size 9 on C are:

$$A_1=\{\theta,1,3\}\times Z_3,$$
$$A_2=\{\theta,2,4\}\times Z_3,$$
$$A_3=\{0,1,2\}\times Z_3,\quad\text{and}$$
$$A_4=\{0,3,4\}\times Z_3.$$

References

[1] H. Beker, On strong tactical decompositions, *J. London Math. Soc.* **16** (1977), 191—196.
[2] H. Beker and F. C. Piper, Some designs which admit strong tactical decompositions, *J. Combinatorial Th. A.*, **22** (1977), 38—42.
[3] R. Bruck, Difference sets in a finite group, *Trans. Amer. Math. Soc.*, **78** (1955), 464—481.
[4] R. Bruck, Construction problems of finite projective planes, *Proc. Conf. Comb., UNC Chapel Hill*, (1967), 426—514.
[5] P. Dembowski, *Finite Geometries*, Springer Verlag, New York/Berlin, (1968).
[6] W. M. Kantor, 2-transitive symmetric designs, *Trans. Amer. Math. Soc.*, **146** (1969), 1—28.
[7] R. Wilson, *personal communication.*

R. D. Baker

Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19711, U. S. A.